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1998 J. Phys. A: Math. Gen. 31 9715

(http://iopscience.iop.org/0305-4470/31/48/008)

J. Phys. A: Math. Gen. **31** (1998) 9715–9727. Printed in the UK PII: S0305-4470(98)95731-6

A new class of solutions to a generalized nonlinear Schrödinger equation

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Received 6 July 1998

Abstract. In this paper we compute new classes of symmetry reduction and associated exact solutions of a generalized nonlinear Schrödinger equation (GNLS), the generalized terms modelling dispersion and scattering. Several authors have obtained symmetry reductions of one-, two- and three-dimensional nonlinear Schrödinger equations; in all cases to date reductions have been based on a *real* new independent variable. In this paper we compute reductions in which the new independent variable is complex. We seek new reductions from a two-dimensional GNLS to a PDE in two independent variables and also reductions to ODEs. Five new classes of reduction are found.

1. Introduction

In this paper we compute a new class of symmetry reductions and associated special, exact solutions of a generalized nonlinear Schrödinger equation:

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + u_{yy} + (\boldsymbol{a}_1 + \mathrm{i}\boldsymbol{a}_2)\cdot(\mathbf{i}\partial_x + \mathbf{j}\partial_y)(u|u|^2) \\ &\qquad + u(\boldsymbol{b}_1 + \mathrm{i}\boldsymbol{b}_2)\cdot(\mathbf{i}\partial_x + \mathbf{j}\partial_y)(|u|^2) + cu|u|^4 + du|u|^2 = 0 \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} \boldsymbol{a}_1 &= a_{11}\mathbf{i} + a_{12}\mathbf{j} & \qquad \boldsymbol{b}_1 &= b_{11}\mathbf{i} + b_{12}\mathbf{j} \\ \boldsymbol{a}_2 &= a_{21}\mathbf{i} + a_{22}\mathbf{j} & \qquad \boldsymbol{b}_2 &= b_{21}\mathbf{i} + b_{22}\mathbf{j} \end{aligned} \tag{1.2}$$

and a_{ij}, b_{ij}, c and d are real constants. This equation is a generalization of the ubiquitous cubic nonlinear Schrödinger equation

$$\mathrm{i}u_t + u_{xx} + du|u|^2 = 0. \tag{1.3}$$

The generalization represents further physical effects such as dispersion and scattering.

Several authors have obtained symmetry reductions of the one-, two- and three-dimensional nonlinear Schrödinger equation. In all cases the new dependent variable is complex but the new dependent variable(s) is (are) real. In this paper we seek reductions in which the new dependent variable(s) is (are) complex.

In many cases the new independent variables in the reductions which have been found are linear in the given spatial variables; such reductions (at least those to ODEs) are of the

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form

$$u(x, y, t) = R(t)p(\xi)\exp\{\mathrm{i}\Phi(x, y, t)\} \tag{1.4a}$$

$$v(x, y, t) = R(t)q(\xi)\exp\{-\mathrm{i}\Phi(x, y, t)\} \tag{1.4b}$$

$$\xi(x, y, t) = (x + \kappa y)\theta(t) + \psi(t) \tag{1.4c}$$

with all functions real, κ real and

$$\theta = R^n \tag{1.5}$$

where n is an integer. (Reductions found to PDEs take a similar form.) Here we suppose that the new independent variable, ξ, is not real; we suppose

$$\xi(x, y, t) = (x + \mathrm{i}y)\theta(t) + \psi_r(t) + \mathrm{i}\psi_i(t) \tag{1.6}$$

where, again, all functions are real. As a result we are no longer constrained by (1.5) and are able to compute large classes of new reductions.

In addition to reductions to ODEs based on (1.6) we consider analogous reductions to PDEs in just two independent variables. We suppose

$$u(x, y, t) = p(\xi, t)\exp\{\mathrm{i}\Phi(x, y, t)\} \qquad v(x, y, t) = q(\xi^*, t)\exp\{-\mathrm{i}\Phi(x, y, t)\} \tag{1.7a}$$

$$\xi(x, y, t) = (x + \mathrm{i}y)\theta + \psi_\mathrm{r}(t) + \mathrm{i}\psi_\mathrm{i}(t). \tag{1.7b}$$

Note that in this case we have been able to set $R = 1$ without loss of generality. (To see that this is indeed without loss of generality consider (2.8) and its accompanying text.)

2. Background

The nonlinear two-dimensional Schrödinger equation was first derived to describe deep water waves (Zakharov 1968) and subsequently has arisen in studies of quantum field theory (Dixon and Tuszyński 1989, Tuszyński and Dixon 1989), weakly nonlinear waves (e.g., Parkes 1987) and optics (e.g., Gagnon and Bélanger 1991).

Special, exact solutions of PDEs are often useful: the solutions may be directly (physically) significant, they may be of use to obtain information about asymptotics of other solutions, or can be used to provide a check on numerical solutions. One powerful, systematic method of determining exact solutions involves the use of *point symmetries*: given (1.1), any transformation of the form

$$\begin{aligned} &x \to \chi(x, y, t, u) \qquad t \to \tau(x, y, t, u) \\ &y \to \eta(x, y, t, u) \qquad u \to \upsilon(x, y, t, u) \end{aligned} \tag{2.1}$$

which leaves the equation *invariant*, i.e. unchanged, is a point symmetry of the equation. Given the point symmetries of a PDE one can find a corresponding transformation to an equation with fewer independent variables, a *reduction*. Alternatively, as we do here, one can seek transformations which reduce a given PDE directly; the corresponding symmetries may then be deduced if required. The standard, *classical*, method of determining point symmetries is due to Lie and is described in several textbooks (Bluman and Kumei 1989, Hill 1992, Olver 1986, Stephani 1990).

Several papers have been published in which reductions and exact solutions of a nonlinear Schrödinger equation are determined which reflect the point symmetries of the equation: Tajiri (1982) found symmetries and associated reductions of the two-dimensional cubic equation ($c = \boldsymbol{a}_1 = \boldsymbol{a}_2 = \boldsymbol{b}_1 = \boldsymbol{b}_2 = 0$); the same symmetries and reductions were later found independently by Ramasan and Debnath (1994); Fushchich and Serov (1987) found some solutions of the three-dimensional equation; Gagnon and co-workers (Gagnon and Winternitz 1988, 1989, 1990, Gagnon *et al* 1988) made a systematic study of the symmetries, associated reductions of the three-dimensional GNLS, and the integrability of the resulting equations in fewer independent variables. All these studies used the classical method due to Lie.

For later reference we give the results obtained for (1.1) from Lie's method (Tajiri 1982, Ramasami and Debnath 1994). The symmetries are

$$\boldsymbol{V}_1 = \partial_t \qquad (t\text{-translation}) \tag{2.2a}$$

$$\boldsymbol{V}_2 = \partial_x \qquad (x\text{-translation}) \tag{2.2b}$$

$$\boldsymbol{V}_3 = \partial_y \qquad (y\text{-translation}) \tag{2.2c}$$

$$\boldsymbol{V}_4 = y\partial_x - x\partial_y \qquad (\text{rotation}) \tag{2.2d}$$

$$\boldsymbol{V}_5 = v\partial_u - u\partial_v \qquad (\text{constant change of phase}) \tag{2.2e}$$

$$\boldsymbol{V}_6 = 2t\partial_t + x\partial_x + y\partial_y - \delta(u\partial_u + v\partial_v) \qquad (\text{scaling}) \tag{2.2f}$$

where $\delta = 1$ if $c = 0$ and $d \neq 0$, and $\delta = \frac{1}{2}$ is $c \neq 0$ and $d = 0$. Also, with $\boldsymbol{a}_{ij} = 0$, we obtain

$$\boldsymbol{V}_7 = 2\mathrm{i}t\partial_x + x(u\partial_u - v\partial_v) \qquad (\text{Galilean boost}) \tag{2.2g}$$

$$\boldsymbol{V}_8 = 2\mathrm{i}t\partial_y + y(u\partial_u - v\partial_v). \qquad (\quad '' \quad '' \quad). \tag{2.2h}$$

The corresponding new independent variables are

$$\xi = \frac{x + \nu_1 t + \nu_2}{\sqrt{Q(t)}} \qquad \zeta = \frac{y + \nu_3 t + \nu_4}{\sqrt{Q(t)}} \tag{2.3}$$

where $Q(t) = \nu_5 t^2 + \nu_6 t + \nu_7$, ν_i, $i = 1, \ldots, 7$, being constants.

More general methods of finding reductions of PDEs exist, the most significant being the so-called non-classical method due to Bluman and Cole (1969), and the direct method due to Clarkson and Kruskal (1989), of which there are, in turn, various extensions and generalizations (Hood 1995, 1997, 1998). The only study using a more general non-classical method, to date, appears to be that by Clarkson (1992) who investigated reductions of the one-dimensional GNLS and showed that the reductions found could readily be extended to higher-dimensional equations. No study of higher-dimensional GNLS equations using a nonclassical method has apparently been made to date. (Clarkson did not study higher-dimensional equations explicitly.)

In this paper we determine several classes of new reduction of the generalized nonlinear Schrödinger equation (1.1), based upon the ansätze (1.6) and (1.7). We follow the Clarkson–Kruskal method, except that the procedures are complicated by the use of a complex new independent variable in addition to complex new dependent variables.

The method is well explained by an example (which is based on appendix B of Clarkson and Kruskal 1989). Suppose we wish to determine reductions of Burgers' equation

$$u_t + uu_x + u_{xx} = 0 \tag{2.4}$$

then one supposes that there exist transformations, $u(x,t) = U(x,t,\omega(\xi(x,t)))$, which yield an ODE in $\omega(\xi)$. It turns out that it is sufficient to consider

$$u(x,t) = \alpha(x,t) + \beta(x,t)\omega(\xi(x,t)) \tag{2.5}$$

(appendix B of Clarkson and Kruskal 1989). Substituting equation (2.5) in (2.4) one finds

$$\beta\xi_x^2\omega'' + (2\beta_x\xi_x + \beta\xi_{xx} + \beta\xi_t + \alpha\beta\xi_x)\omega' + (\beta_{xx} + \beta_t + \alpha\beta_x + \alpha_x\beta)\omega$$
$$+\beta^2\xi_x\omega\omega' + \beta\beta_x\omega^2 + \alpha_t + \alpha\alpha_x + \alpha_{xx} = 0 \qquad (2.6)$$

and requiring that this be an ODE one then obtains a system of PDEs for the unknown functions, α, β and ξ, viz.

$$\beta\xi_x^2\Gamma_1(\xi) = 2\beta_x\xi_x + \beta\xi_{xx} + \beta\xi_t + \alpha\beta\xi_x \qquad (2.7a)$$

$$\beta\xi_x^2\Gamma_2(\xi) = \beta_{xx} + \beta_t + \alpha\beta_x + \alpha_x\beta \qquad (2.7b)$$

$$\beta\xi_x^2\Gamma_3(\xi) = \beta^2\xi_x \qquad (2.7c)$$

$$\beta\xi_x^2\Gamma_4(\xi) = \beta\beta_x \qquad (2.7d)$$

$$\beta\xi_x^2\Gamma_(5\xi) = \alpha_t + \alpha\alpha_x + \alpha_{xx} \qquad (2.7e)$$

where $\Gamma_i(\xi)$, $i = 1, \ldots, 5$, are to be determined. Solution of the system yields the desired reductions.

Any particular reduction is in fact a class of reductions equivalent through a transformation of the new variables. One can make use of this by redefinitions of these variables during the computation:

1. For example, any reduction which leads to

$$dR^3(t)|p|^2p + \mathrm{i}R(t)\dot{p} - \mathrm{i}f(t)R(t)p_\xi + \mathrm{i}\dot{R}(t)p + g(t)R(t)p = 0 \qquad (2.8)$$

   is equivalent to one in which $R = 1$ through a rescaling of $p \to R^{-1}p$. This is why we may set $R = 1$ without loss of generality in ansatz (1.7).
2. One can also redefine the new independent variable in a similar way, however, since we have already chosen the form in this paper (cf equation (1.6)) then we do not use this freedom here.

Many reductions of the GNLS found to date are of the form

$$u(x, y, t) = R(t)p(\xi)\exp\{\mathrm{i}\Phi(x, y, t)\} \qquad (2.9a)$$

$$\xi(x, y, t) = \boldsymbol{n}\cdot\boldsymbol{x}\theta(t) + \psi(t) \qquad (2.9b)$$

(in one, two or three dimensions, and by both classical and non-classical methods). In each one finds that θ and R must satisfy (1.5); this is a particularly restrictive condition. Substituting equations (2.9) in (3.1a) one finds a result of the form

$$(n_1^2 + n_2^2 + n_3^2)R\theta^2p'' + R^3\theta(2a_{11} + \cdots)pqp' + R^3\theta(a_{11} - \cdots)p^2q'$$
$$+ R^3(d + \cdots)p^2q + (\cdots)p' + (\cdots)p = 0. \qquad (2.10)$$

Balancing the coefficient of p'' with the other coefficients leads to (1.5). In this paper we use ansätze which circumvent (1.5): the new independent variable is complex. (To the author's knowledge only real new independent variables have been previously considered.)

Notation. Unless otherwise stated the following notation is used in sections 3 and 4:

1. λ_i, $i = 1, 2, \ldots$ are *real* constants, for example constants of integration, which remain arbitrary.
2. Γ_i, $i = 1, 2, \ldots$ are functions obtained in balancing coefficients of products of powers of derivatives of new dependent variables; these are to be determined. Often it turns out that these are necessarily constant. In this case Γ_i is replaced with γ_i. (In addition, in section 4 we use the lower case when a balance-function is necessarily a function of t alone.)

3. New reductions to ODEs

In this section we compute reductions of (1.1) by using the ansatz (1.6). Two terms in (1.1) are non-analytic. To overcome this problem, we define $v(x, y, t)$ to be the complex conjugate of $u(x, y, t)$ and so write (1.1) as the system

$$\mathrm{i}u_t + u_{xx} + u_{yy} + (\boldsymbol{a}_1 + \mathrm{i}\boldsymbol{a}_2) \cdot (\mathbf{i}\partial_x + \mathbf{j}\partial_y)(u^2 v) + u(\boldsymbol{b}_1 + \mathrm{i}\boldsymbol{b}_2) \cdot (\mathbf{i}\partial_x + \mathbf{j}\partial_y)(uv) + cu^3v^2 + du^2v = 0 \tag{3.1a}$$

$$-\mathrm{i}v_t + v_{xx} + v_{yy} + (\boldsymbol{a}_1 - \mathrm{i}\boldsymbol{a}_2) \cdot (\mathbf{i}\partial_x + \mathbf{j}\partial_y)(uv^2) + v(\boldsymbol{b}_1 - \mathrm{i}\boldsymbol{b}_2) \cdot (\mathbf{i}\partial_x + \mathbf{j}\partial_y)(uv) + cu^2v^3 + duv^2 = 0. \tag{3.1b}$$

Remarks.

1. Note that since ξ is complex we require $q(\xi)$ to be the complex-conjugate of $p(\xi)$ throughout the complex plane, rather than just along the real line.
2. When balancing coefficients of products of powers of derivatives of p and q, in general, each resulting equation is complex and is therefore equivalent to two real equations.

Substituting equation (1.6) in (3.1a) and grouping terms we find

$$\begin{aligned} cR^5p^3q^2 &+ pqp'\{R^3\theta[2a_{11} - 2a_{22} + b_{11} - b_{22} + \mathrm{i}(2a_{21} + 2a_{12} + b_{21} + b_{12})]\} \\ &+ p^2q'\{R^3\theta[a_{11} - a_{22} + b_{11} - b_{22} + \mathrm{i}(a_{21} + a_{12} + b_{21} + b_{12})]\} \\ &+ p^2q\{R^3(d + [a_{11} + b_{11} + \mathrm{i}(a_{21} + b_{21})]\Phi_x \\ &+ [a_{12} + b_{12} + \mathrm{i}(a_{22} + b_{22})]\Phi_y)\} + p'\{\mathrm{i}R\xi_t + 2\mathrm{i}R\theta(\Phi_x + \mathrm{i}\Phi_y)\} \\ &+ p\{\mathrm{i}\dot{R} + \mathrm{i}R(\Phi_{xx} + \Phi_{yy}) - R(\dot{\Phi} + \Phi_x^2 + \Phi_y^2)\} = 0. \end{aligned} \tag{3.2}$$

Focusing on the coefficients of p^3q^2, pqp' and p^2q' we find that there are three cases to consider. First, $\dot{\theta} = \dot{R} = 0$ with $c \neq 0$ and at least one of the coefficients of pqp' and p^2q' non-zero. Second, $c \neq 0$ and the coefficients of pqp' and p^2q both equal to zero. Third, $c = 0$ and the coefficients of both pqp' and p^2q' zero. We consider each of these cases in turn in the follwoing subsections. Since the computation is similar for each subsection, we give the details leading to reductions only in subsection 3.2; in the other cases we simply quote the results. (One might suppose two other cases exist: first, $c \neq 0$ and at least one of the coefficients of pqp' and p^2q non-zero, with $\theta(t) = R^2(t)$; second, $c = 0$ and at least one of the coefficients of pqp' and p^2q non-zero with $\dot{\theta} \neq 0$. It turns out that there are no reductions in either case.)

3.1. $c \neq 0$, at least one of the coefficients of pqp' and p^2q' non-zero

In this section we consider the first case, i.e. $\dot{\theta} = \dot{R} = 0$. There is just one reduction.

Reduction 1. For any values of a_{ij}, b_{ij}, c and d, then

$$u(x,y,t) = p(\xi)\exp\{\mathrm{i}\Phi(x,y,t)\} \qquad v(x,y,t) = q(\xi^*)\exp\{-\mathrm{i}\Phi(x,y,t)\} \tag{3.3a}$$

$$\Phi(x,y,t) = -\tfrac{1}{2}\lambda_1 x - \tfrac{1}{2}(\lambda_3 - \gamma_{1\mathrm{i}})y - \tfrac{1}{4}(\lambda_1^2 + \lambda_3^2) - \gamma_{40\mathrm{r}} \tag{3.3b}$$

$$\xi(x,y,t) = x + \mathrm{i}y + \lambda_1 t + \lambda_2 + \mathrm{i}(\lambda_3 t + \lambda_4) \tag{3.3c}$$

reduces the two-dimensional GNLS to the system

$$\begin{aligned} cp^3q^2 &+ \big\{2a_{11} - 2a_{22} + b_{11} - b_{22} + \mathrm{i}(2a_{21} + 2a_{12} + b_{21} + b_{12})\big\}pqp' \\ &+ \big\{a_{11} - a_{22} + b_{11} - b_{22} + \mathrm{i}(a_{21} + a_{12} + b_{21} + b_{12})\big\}p^2q' \\ &+ \big\{d - \tfrac{1}{2}\lambda_1[(a_{11} + b_{11}) + \mathrm{i}(a_{21} + b_{21})] \\ &- \tfrac{1}{2}(\lambda_3 - \gamma_{1\mathrm{i}})[a_{12} + b_{12} + \mathrm{i}(a_{22} + b_{22})]\big\}p^2q - \gamma_{1\mathrm{i}}p' + \gamma_{40\mathrm{r}}p = 0 \end{aligned} \tag{3.4a}$$

$$\text{c.c.} = 0. \tag{3.4b}$$

To integrate (3.4) write $p = r\exp(\mathrm{i}\sigma)$, $q = r\exp(-\mathrm{i}\sigma)$, and take real and imaginary parts yielding

$$\begin{aligned} cr^5 &+ \big\{3a_{11} - 3a_{22} + 2b_{11} + 2b_{22}\big\}r^2r' - \big\{a_{21} + a_{12}\big\}r^3\sigma' + \big\{d - \tfrac{1}{2}\lambda_1(a_{11} + b_{11}) \\ &- \tfrac{1}{2}(\lambda_3 - \gamma_{1\mathrm{i}})(a_{12} + b_{12}\big\}r^3 - \gamma_{1\mathrm{i}}r' + \gamma_{40}r = 0 \end{aligned} \tag{3.5a}$$

$$\begin{aligned} \big\{3a_{21} &+ 3a_{12} + 2b_{21} + 2b_{12}\big\}r^2r' + \big\{a_{11} - a_{22}\big\}r^3\sigma' \\ &- \tfrac{1}{2}\big\{\lambda_1(a_{21} + b_{21}) + (\lambda_3 - \gamma_{1\mathrm{i}})(a_{22} + b_{22})\big\}r^3 - \gamma_{1\mathrm{i}}r\sigma' = 0. \end{aligned} \tag{3.5b}$$

This system for r and σ is integrable in terms of quadratures; there are several cases depending upon the value of the (constant) coefficients within the system.

3.2. $c \neq 0$, coefficients of both pqp' and p^2q' zero

Given that the coefficients of both pqp' and p^2q' are equal to zero, after a little algebra we find

$$\begin{aligned} a_{11} - a_{22} &= 0 \qquad & a_{21} + a_{12} &= 0 \\ b_{11} - b_{22} &= 0 \qquad & b_{21} + b_{12} &= 0. \end{aligned} \tag{3.6}$$

Equation (3.2) becomes

$$\begin{aligned} cR^5p^3q^2 &+ p'\big\{\mathrm{i}R\xi_t + 2\mathrm{i}R\theta(\Phi_x + \mathrm{i}\Phi_y)\big\} + p^2q\big\{R^3\big(d + [a_{11} + b_{11} + \mathrm{i}(a_{21} + b_{21})]\Phi_x \\ &+ [a_{12} + b_{12} + \mathrm{i}(a_{22} + b_{22})]\Phi_y\big)\big\} + p\big\{\mathrm{i}\dot{R} + \mathrm{i}R(\Phi_{xx} + \Phi_{yy}) \\ &- R(\dot{\Phi} + \Phi_x^2 + \Phi_y^2)\big\} = 0 \end{aligned} \tag{3.7}$$

and then balancing coefficients of powers of products of derivatives of p and q we find the determining system

$$\xi_t + 2\theta(\Phi_x + \mathrm{i}\Phi_y) = R^4\Gamma_{1,\xi} \tag{3.8a}$$

$$\dot{R} + R(\Phi_{xx} + \Phi_{yy}) = R^5\Gamma_2 \tag{3.8b}$$

$$-\dot{\Phi} - \Phi_x^2 - \Phi_y^2 = R^4\Gamma_3 \tag{3.8c}$$

$$d + [a_{11} + b_{11} + \mathrm{i}(a_{21} + b_{21})]\Phi_x + [a_{12} + b_{12} + \mathrm{i}(a_{22} + b_{22})]\Phi_y = R^2\Gamma_4. \tag{3.8d}$$

Remarks.

1. We have introduced $\Gamma_{1,\xi}$ rather than Γ_1 purely for notational convenience.
2. $\Gamma_{1,\xi}$ is the coefficient of $\mathrm{i}p'$, not p'.
3. Equations (3.8*a*) and (3.8*b*) are complex and therefore each represents two real equations, whilst (3.8*b*) and (3.8*c*) are real equations, from the imaginary and real parts of the coefficent of p.

Adding equations (3.8*a*) to its complex conjugate we obtain a quadrature for $\Phi(x, y, t)$ and integrating this with respect to x we obtain

$$\Phi(x, y, t) = \frac{R^4}{4\theta^2}(\Gamma_1 + \Gamma_1^*) - \frac{x^2\dot{\theta}}{4\theta} - \frac{x\dot{\psi}_r}{2\theta} + \phi_1(y, t) \tag{3.9}$$

where $\phi_1(y, t)$ is a function of integration, to be determined, and Γ_1^* is the complex conjugate of Γ_1. Substituting back in (3.8*a*) we find

$$\mathrm{i}(y\dot{\theta} + \dot{\psi}_{\mathrm{i}} + 2\theta\phi_{1,y}) = R^4\Gamma_1(\xi). \tag{3.10}$$

The right-hand side is independent of x and has no real part, so $\Gamma_{1,\xi}$ is necessarily constant, $\mathrm{i}\gamma_{1\mathrm{i}}$, say. Now integrating (3.10) we obtain

$$\phi_1(y, t) = -\frac{y^2\dot{\theta}}{4\theta} - \frac{y}{2\theta}(\dot{\psi}_{\mathrm{i}} - \gamma_{1\mathrm{i}}R^4) + \phi_{10}(t) \tag{3.11}$$

where $\phi_1(t)$ is a function of integration. Collecting results we have

$$\Phi(x, y, t) = -\frac{\dot{\theta}}{4\theta}\left(x^2 + y^2\right) - \frac{x\dot{\psi}_{\mathrm{r}}}{2\theta} - \frac{y}{2\theta}\left(\dot{\psi}_{\mathrm{i}} - \gamma_{1\mathrm{i}}R^4\right) + \phi_{10}(t). \tag{3.12}$$

Given (3.12), the left-hand side of (3.8*b*) is independent of x and y, and is also real, so that Γ_2 is necessarily constant and real, $\gamma_{2\mathrm{r}}$, say, and we have

$$\dot{R} - \frac{\dot{\theta}}{\theta}R = \gamma_{2\mathrm{r}}R^5 \tag{3.13}$$

a Bernoulli equation for $R(t)$, given $\theta(t)$.

Next, substituting (3.12) in (3.8*c*) we see that the left-hand side is quadratic in both x and y so, without loss of generality, we suppose that $\Gamma_3(\xi) = \gamma_{32}\xi^2 + \gamma_{31}\xi + \gamma_{30}$. Then equating like coefficients of products of powers of x and y, and taking real and imaginary parts, we find $\gamma_{32} = \gamma_{31} = 0$, γ_{30} is real, $\gamma_{30\mathrm{r}}$, say, and

$$\theta\ddot{\theta} - 2\dot{\theta}^2 = 0 \tag{3.14a}$$

$$\theta\ddot{\psi}_{\mathrm{r}} - 2\dot{\theta}\dot{\psi}_{\mathrm{r}} = 0 \tag{3.14b}$$

$$\theta\ddot{\psi}_{\mathrm{i}} - 2\dot{\theta}\dot{\psi}_{\mathrm{i}} - 4\gamma_{1\mathrm{i}}\theta R^3\dot{R} + 2\gamma_{1\mathrm{i}}\dot{\theta}R^4 = 0 \tag{3.14c}$$

and

$$\phi_{10}(t) = \int \left\{ \gamma_{30\mathrm{r}} + \frac{\dot{\psi}_\mathrm{r}^2}{4\theta^2} + \frac{1}{4\theta^2}(\dot{\psi}_\mathrm{i}^2 + \gamma_{1\mathrm{i}} R^4)^2 \right\} \mathrm{d}t. \tag{3.15}$$

It remains to check consistency with (3.8*d*). Substituting equation (3.12) in (3.8*d*) we see that Γ_4 must be linear, $(\gamma_{41\mathrm{r}} + \mathrm{i}\gamma_{41\mathrm{i}})\xi + \gamma_{40\mathrm{r}} + \gamma_{40\mathrm{i}}$ say. Then equating coefficients of products of powers of x and y, and taking real and imaginary parts we find that $\gamma_{41\mathrm{r}} = \gamma_{41\mathrm{i}} = \gamma_{40\mathrm{i}} = 0$ and consequently

$$a_{11} + b_{11} = a_{12} + b_{12} = 0 \tag{3.16}$$

and $d = \gamma_{40\mathrm{r}} R^2$, i.e. R is necessarily constant if d is non-zero. This is inconsistent with (3.14*a*) and (3.13), so $d = 0$. Similarly we find that

$$a_{21} + b_{21} = a_{22} + b_{22} = 0. \tag{3.17}$$

(Note that equations (3.6), (3.16) and (3.17) do *not* mean that $a_{ij} = 0$ and/or $b_{ij} = 0$, for all $i,\ j$!)

Neglecting constant solutions of (3.14*a*), which are considered in subsection 3.1, we have obtained one reduction of (3.1).

Reduction 2. Given equations (3.6), (3.16) and (3.17) (conditions on a_{ij} and b_{ij}, which, as a special case are satisfied by $a_{ij} = b_{ij} = 0$ for all $i,\ j$), and provided $d = 0$ then

$$u(x, y, t) = R(t) p(\xi) \exp\{\mathrm{i}\Phi(x, y, t)\} \qquad v(x, y, t) = R(t) q(\xi^*) \exp\{-\mathrm{i}\Phi(x, y, t)\} \tag{3.18a}$$

$$\Phi(x, y, t) = \frac{x^2 + y^2}{4t} - \tfrac{1}{2}\lambda_1 x t - \tfrac{1}{2} y t \left(\dot{\psi}_\mathrm{i} + \gamma_{1\mathrm{i}} R^4(t)\right) + \phi_{10}(t) \tag{3.18b}$$

$$\xi(x, y, t) = \frac{x + \mathrm{i}y}{t} - \frac{\lambda_1}{t} + \lambda_2 + \mathrm{i}\psi_\mathrm{i}(t) \tag{3.18c}$$

where

$$R(t) = \left(\lambda_3 t^4 + \tfrac{4}{3}\gamma_{2\mathrm{r}} t\right)^{-1/4} \tag{3.18d}$$

$$\psi_\mathrm{i}(t) = \gamma_{1\mathrm{i}} \int R^4(t)\, \mathrm{d}t - \frac{\lambda_4}{t} + \lambda_5 \tag{3.18e}$$

and where p and q satisfy

$$c p^3 q^2 - \gamma_{1\mathrm{i}} p' + (\gamma_{30\mathrm{r}} + \mathrm{i}\gamma_{2\mathrm{r}}) p = 0 \tag{3.19a}$$

$$c p^2 q^3 - \gamma_{1\mathrm{i}} q' + (\gamma_{30\mathrm{r}} - \mathrm{i}\gamma_{2\mathrm{r}}) q = 0. \tag{3.19b}$$

To integrate (3.19) we write $p = r\exp(\mathrm{i}\sigma)$, $q = r\exp(-\mathrm{i}\sigma)$, and substituting and taking real and imaginary parts we find

$$c r^5 - \gamma_{1\mathrm{i}} r' + \gamma_{30\mathrm{r}} r = 0 \tag{3.20a}$$

$$-\gamma_{1\mathrm{i}} \sigma' + \gamma_{2\mathrm{r}} = 0 \tag{3.20b}$$

from which it is easy to determine both r and σ. We find that

$$\int^r \frac{\gamma_{1\mathrm{i}}\, \mathrm{d}r_1}{c r_1^5 + \gamma_{30\mathrm{r}} r_1} = \xi + \lambda_6 \qquad \sigma = \frac{\gamma_{2\mathrm{r}}}{\gamma_{1\mathrm{i}}} \xi + \lambda_7. \tag{3.21}$$

3.3. $c = 0$, coefficients of pqp' and p^2q' zero

We obtain just one reduction.

Reduction 3. Provided $c = 0$ and

$$a_{11} - a_{22} = a_{21} + a_{12} = b_{11} - b_{22} = b_{21} + b_{12}$$
$$= a_{11} + b_{11} = a_{12} + b_{12} = a_{21} + b_{21} = a_{22} + b_{22} = 0.$$

then

$$u(x, y, t) = R(t)p(\xi)\exp\{\mathrm{i}\Phi(x, y, t)\} \qquad v(x, y, t) = R(t)q(\xi^*)\exp\{-\mathrm{i}\Phi(x, y, t)\} \tag{3.22a}$$

$$\Phi(x, y, t) = \frac{x^2 + y^2}{4t} - \tfrac{1}{2}\lambda_1 xt - \tfrac{1}{2}yt\left(\dot{\psi}_\mathrm{i} + \gamma_{1\mathrm{i}}R^4(t)\right) + \phi_{10}(t) \tag{3.22b}$$

$$\xi(x, y, t) = \frac{x + \mathrm{i}y}{t} - \frac{\lambda_1}{t} + \lambda_2 + \mathrm{i}\psi_\mathrm{i}(t) \tag{3.22c}$$

where

$$R(t) = \left(\lambda_3 t^2 + 2\gamma_{2\mathrm{r}}t\right)^{-1/2} \tag{3.22d}$$

$$\psi_\mathrm{i}(t) = \gamma_{1\mathrm{i}}\int R^2(t)\,\mathrm{d}t - \frac{\lambda_4}{t} + \lambda_5 \tag{3.22e}$$

and where p and q satisfy

$$dp^2q - \gamma_{1\mathrm{i}}p' + (\gamma_{30\mathrm{r}} + \mathrm{i}\gamma_{2r})p = 0 \tag{3.23a}$$

$$dpq^2 - \gamma_{1\mathrm{i}}q' + (\gamma_{30\mathrm{r}} - \mathrm{i}\gamma_{2r})q = 0. \tag{3.23b}$$

As before, we write $p = r\exp(\mathrm{i}\sigma)$, $q = r\exp(-\mathrm{i}\sigma)$; after substituting, and taking real and imaginary parts we find

$$\int^r \frac{\gamma_{1\mathrm{i}}\,\mathrm{d}r_1}{dr_1^3 + \gamma_{30\mathrm{r}}r_1} = \xi + \lambda_6 \qquad \sigma = \frac{\gamma_{2\mathrm{r}}}{\gamma_{1\mathrm{i}}}\xi + \lambda_7. \tag{3.24}$$

4. New reductions to PDEs

In this section we look for reductions of our generalized nonlinear Schrödinger equation (3.1) to a PDE, by using (1.7). Substituting in (3.1) and grouping terms we find

$$\begin{aligned} cp^3q^2 + pqp'&\left\{\theta[2a_{11} - 2a_{22} + b_{11} - b_{22} + \mathrm{i}(2a_{21} + 2a_{12} + b_{21} + b_{12})]\right\} \\ &+ p^2q'\left\{\theta[a_{11} - a_{22} + b_{11} - b_{22} + \mathrm{i}(a_{21} + a_{12} + b_{21} + b_{12})]\right\} \\ &+ p^2q\left\{\left(d + [a_{11} + b_{11} + \mathrm{i}(a_{21} + b_{21})]\Phi_x + [a_{12} + b_{12} + \mathrm{i}(a_{22} + b_{22})]\Phi_y\right)\right\} \\ &+ p'\left\{\mathrm{i}\xi_t + 2\mathrm{i}\theta(\Phi_x + \mathrm{i}\Phi_y)\right\} + \mathrm{i}\dot{p} \\ &+ p\left\{\mathrm{i}(\Phi_{xx} + \Phi_{yy}) - \dot{\Phi} - \Phi_x^2 - \Phi_y^2)\right\} = 0. \end{aligned} \tag{4.1}$$

When looking for reductions to ODEs there were several cases to consider owing to the necessary balance of the coefficients of p^3q^2, pqp' and p^2q' (which were functions of t, only); here t is one of the new independent variables so this complication does not arise: the coefficients of pqp' and p^2q' are 'automatically' balanced, as is the coefficient of $\dot{p}$.

It remains to balance the coefficients of p^2q, p_ξ and p. We obtain the determining system

$$(x+\mathrm{i}y)\dot{\theta}+\dot{\psi}_{\mathrm{r}}+\mathrm{i}\dot{\psi}_{\mathrm{i}}+2\theta(\Phi_x+\mathrm{i}\Phi_y)=\Gamma_{1,\xi}(\xi,t) \tag{4.2a}$$

$$\Phi_{xx}+\Phi_{yy}=\Gamma_2(\xi,t) \tag{4.2b}$$

$$-\dot{\Phi}-\Phi_x^2-\Phi_y^2=\Gamma_3(\xi,t) \tag{4.2c}$$

$$d+[a_{11}+b_{11}+\mathrm{i}(a_{21}+b_{21})]\Phi_x+[a_{12}+b_{12}+\mathrm{i}(a_{22}+b_{22})]\Phi_y=\Gamma_4(\xi,t) \tag{4.2d}$$

where $\Gamma_1,\ldots,\Gamma_4$ are to be determined.

Remarks.

1. Equations (4.2a) and (4.2d) are complex and therefore each represents two real equations; equations (4.2b) and (4.2c) come from the imaginary and real parts, respectively, of the coefficient of p (and are therefore real).
2. We have introduced $\Gamma_{1,\xi}$ rather than Γ_1 purely for notational convenience.

Adding equation (4.2a) to its complex conjugate yields a quadrature for Φ_x and integrating we obtain

$$\Phi(x,y,t)=\frac{\Gamma_1+\Gamma_1^*}{4\theta^2}-\frac{x^2\dot{\theta}}{4\theta}-\frac{\dot{\psi}_{\mathrm{r}}x}{2\theta}+\phi_1(y,t) \tag{4.3}$$

where $\phi_1(y,t)$ is a function of integration, to be determined. Substituting this result back in (4.2a) we find

$$\mathrm{i}(y\dot{\theta}+\dot{\psi}_{\mathrm{i}}+2\theta\phi_{1,y})=\Gamma_{1,\xi}. \tag{4.4}$$

The left-hand side of this is independent of x and purely imaginary, so $\Gamma_{1,\xi}$ is necessarily a purely-imaginary function of t alone, $\mathrm{i}\gamma_{1\mathrm{i}}(t)$, say. Integrating equation (4.4) we obtain

$$\phi_1(y,t)=-\frac{y^2\dot{\theta}}{4\theta}-\frac{y(\dot{\psi}_{\mathrm{i}}-\gamma_{1\mathrm{i}}(t))}{2\theta}. \tag{4.5}$$

Note that we have taken the function (of t) of integration to be zero without loss of generality, through a rescaling of p (freedom I). So, collecting results,

$$\Phi(x,y,t)=-\frac{(x^2+y^2)\dot{\theta}}{4\theta}-\frac{x\dot{\psi}_{\mathrm{r}}}{2\theta}-\frac{y(\dot{\psi}_{\mathrm{i}}-\gamma_{1\mathrm{i}}(t))}{2\theta}. \tag{4.6}$$

Substituting equation (4.6) in (4.2b) we find that the left-hand side is independent of both x and y, and real, so Γ_2 is necessarily a real function of t, $\gamma_2(t)$, say, where

$$-\frac{\dot{\theta}}{\theta}=\gamma_2(t). \tag{4.7}$$

Next, substituting equations (4.6) in (4.2c) we find that the left-hand side is quadratic in both x and y; there is no bilinear term so that the right-hand side must be linear in ξ, $(\gamma_{31\mathrm{r}}(t)+\mathrm{i}\gamma_{31\mathrm{i}}(t))\xi+\gamma_{30\mathrm{r}}(t)+\mathrm{i}\gamma_{30\mathrm{i}}(t)$, say. Then equating coefficients of like powers of x and y, and taking real and imaginary parts we find

$$\theta\ddot{\theta}-2\dot{\theta}^2=0 \tag{4.8a}$$

$$\theta\ddot{\psi}_{\mathrm{r}}-2\dot{\theta}\dot{\psi}_{\mathrm{r}}=0 \tag{4.8b}$$

$$\frac{\ddot{\psi}_{\mathrm{i}}}{2\theta}-\frac{\dot{\psi}_{\mathrm{i}}\dot{\theta}}{\theta^2}+\frac{\dot{\theta}\gamma_{1\mathrm{i}}(t)}{\theta^2}-\frac{\dot{\gamma}_{1\mathrm{i}}}{2\theta}=0 \tag{4.8c}$$

$$-\dot{\psi}_{\mathrm{r}}^2-\left(\dot{\psi}_{\mathrm{i}}-\gamma_{1\mathrm{i}}(t)\right)^2=4\theta^2\gamma_{30\mathrm{r}}(t) \tag{4.8d}$$

together with

$$\gamma_{31\mathrm{r}} = \gamma_{31\mathrm{i}} = \gamma_{30\mathrm{i}} = 0. \tag{4.8e}$$

It remains to check consistency with (4.2d). Substituting equation (4.6) in (4.2d) we find that the left-hand side is linear in both x and y and therefore Γ_4 is linear in ξ, $(\gamma_{41\mathrm{r}}(t) + \mathrm{i}\gamma_{41\mathrm{i}}(t))\xi + \gamma_{40\mathrm{r}}(t) + \mathrm{i}\gamma_{40\mathrm{i}}(t)$, say. Equating coefficients of like powers of x and y and taking real and imaginary parts we find

$$(a_{11} + b_{11})\dot{\theta} = -2\gamma_{41\mathrm{r}}(t)\theta^2 \tag{4.9a}$$

$$(a_{21} + b_{21})\dot{\theta} = -2\gamma_{41\mathrm{i}}(t)\theta^2 \tag{4.9b}$$

$$(a_{12} + b_{12})\dot{\theta} = 2\gamma_{41\mathrm{i}}(t)\theta^2 \tag{4.9c}$$

$$(a_{22} + b_{22})\dot{\theta} = -2\gamma_{41\mathrm{r}}(t)\theta^2 \tag{4.9d}$$

$$2d\theta - (a_{11} + b_{11})\dot{\psi}_\mathrm{r} - (a_{12} + b_{12})\big(\dot{\psi}_\mathrm{i} - \gamma_{1\mathrm{i}}(t)\big) = 2\theta\left\{\gamma_{41\mathrm{r}}\psi_\mathrm{r} - \gamma_{41\mathrm{i}}\psi_\mathrm{i} + \gamma_{40\mathrm{r}}(t)\right\} \tag{4.9e}$$

$$-(a_{21} + b_{21})\dot{\psi}_\mathrm{r} - (a_{22} + b_{22})\big(\dot{\psi}_\mathrm{i} - \gamma_{1\mathrm{i}}(t)\big) = 2\theta\left\{\gamma_{41\mathrm{i}}\psi_\mathrm{r} + \gamma_{41\mathrm{r}}\psi_\mathrm{i} + \gamma_{40\mathrm{i}}(t)\right\}. \tag{4.9f}$$

There are two cases to consider: first, if $\dot{\theta} = 0$ then there are no conditions on a_{ij} and b_{ij}; otherwise, if $\dot{\theta} \neq 0$, then we require $a_{11} + b_{11} - a_{22} - b_{22} + \mathrm{i}(a_{21} + b_{21} + a_{12} + b_{12}) = 0$. So, collecting results we have found two classes of reductions.

Reduction 4. The following reduction holds for all values of a_{ij}, b_{ij}, c and d:

$$u(x, y, t) = p(\xi, t)\exp\{\mathrm{i}\Phi(x, y, t)\} \qquad v(x, y, t) = q(\xi^*, t)\exp\{-\mathrm{i}\Phi(x, y, t)\} \tag{4.10a}$$

$$\Phi(x, y, t) = -\tfrac{1}{2}\lambda_1 x - \tfrac{1}{2}\lambda_3 y \tag{4.10b}$$

$$\xi(x, y, t) = x + \mathrm{i}y + \lambda_1 t + \lambda_2 + \mathrm{i}\left\{\int^t \gamma_{1\mathrm{i}}(t_1)\,\mathrm{d}t_1 + \lambda_3 t + \lambda_4\right\} \tag{4.10c}$$

where p and q satisfy

$$\begin{aligned} cp^3q^2 &+ \{2a_{11} - 2a_{22} + b_{11} - b_{22} + \mathrm{i}(2a_{21} + 2a_{12} + b_{21} + b_{12})\}\, pqp' \\ &+ \{a_{11} - a_{22} + b_{11} - b_{22} + \mathrm{i}(a_{21} + a_{12} + b_{21} + b_{12})\}\, p^2q' \\ &+ (\gamma_{40\mathrm{r}} + \mathrm{i}\gamma_{40\mathrm{i}})p^2q - \gamma_{1\mathrm{i}}(t)p' + \mathrm{i}\dot{p} + \gamma_{30\mathrm{r}}p = 0 \end{aligned} \tag{4.11a}$$

$$\text{c.c.} = 0 \tag{4.11b}$$

where

$$\gamma_{30\mathrm{r}} = -\tfrac{1}{4}\lambda_1^2 - \tfrac{1}{4}\lambda_3^2 \tag{4.12a}$$

$$\gamma_{40\mathrm{r}} = d - \tfrac{1}{2}(a_{11} + b_{11})\lambda_1 - \tfrac{1}{4}(a_{12} + b_{12})\lambda_3 \tag{4.12b}$$

$$\gamma_{40\mathrm{i}} = -\tfrac{1}{2}(a_{21} + b_{21})\lambda_1 - (a_{22} + b_{22})\tfrac{1}{2}\lambda_3. \tag{4.12c}$$

Note that $\gamma_{1\mathrm{i}}(t)$ remains an arbitrary function of t which cannot be scaled or translated away. The method of (symbolic) integration of (4.11) is by no means obvious: with the coefficient of p^2q' zero then for some functions $\gamma_{1\mathrm{i}}(t)$, equation (4.11a), may be integrated by characteristics, but this does not appear to be so for general a_{ij}, b_{ij} and $\gamma_{1\mathrm{i}}(t)$.

Reduction 5. Provided $a_{11}+b_{11}-a_{22}-b_{22}+\mathrm{i}(a_{21}+b_{21}+a_{12}+b_{12})=0$ (the coefficients of pqp' and p^2q' are zero as a consequence) then

$$u(x,y,t)=p(\xi,t)\exp\{\mathrm{i}\Phi(x,y,t)\} \qquad v(x,y,t)=q(\xi^*,t)\exp\{-\mathrm{i}\Phi(x,y,t)\} \tag{4.13a}$$

$$\Phi(x,y,t)=\frac{x^2+y^2}{4t}+\frac{\lambda_3 x}{t}-\tfrac{1}{2}yt\big(\dot\psi_{\mathrm{i}}-\gamma_{1\mathrm{i}}(t)\big) \tag{4.13b}$$

$$\xi(x,y,t)=\frac{x+\mathrm{i}y}{t}+\lambda_1-\frac{\lambda_2}{t}+\mathrm{i}\psi_{\mathrm{i}}(t) \tag{4.13c}$$

where $\psi_{\mathrm{i}}(t)$ is given by (4.8*c*), and where p and q satisfy

$$cp^3q^2+\left\{\gamma_{40\mathrm{r}}(t)+\mathrm{i}\gamma_{40\mathrm{i}}(t)+\tfrac{1}{2}(a_{11}+b_{11})+\tfrac{1}{2}\mathrm{i}(a_{22}+b_{22})\right\}p^2q$$
$$-\gamma_{1\mathrm{i}}(t)p'+\mathrm{i}\dot p+\frac{\mathrm{i}}{t}p+\gamma_{30\mathrm{r}}(t)p=0 \tag{4.14}$$

in which $\gamma_{30\mathrm{r}}(t)$ is given by (4.8*d*), $\gamma_{40\mathrm{r}}(t)$ is given by (4.9*e*) and $\gamma_{40\mathrm{i}}(t)$ by (4.9*f*). As before $\gamma_{1\mathrm{i}}(t)$ remains an arbitrary function of t. In cases where equation (4.11) is transformable to an equation of constant coefficients and in a small number of other cases equation (4.11) is integrable by means of characteristics. In other cases the means of (symbolic) integration is not obvious.

5. Discussion

In this paper we have computed new classes of reduction and associated (special) exact solution of a generalized nonlinear Schrödinger equation (1.1). The new classes are obtained by allowing the new independent variable to by complex. In all previously computed reductions of the GNLS the new independent variable has been real.

In section 3 we computed reductions directly from the two-dimensional GNLS to an ODE, i.e. from an equation in three independent variables to an equation in one; three new classes of reduction were found. In section 4 we computed reductions from the two-dimensional GNLS to a PDE in just two independent variables. We computed two new classes of reductions. One of these new classes includes an arbitrary function in t (which cannot be scaled or otherwise transformed away). To the author's knowledge no previously computed reduction of the GNLS has included an arbitrary function.

Finally, it is a simple matter to extend the results obtained here to three (or higher) dimensions. The necessary requirement is a vanishing coefficient of p'', so for an ansatz in which the new independent variable is given by

$$\xi(x,y,z,t)=(x+\kappa_1 y+\kappa_2 z)\theta(t)+\psi_{\mathrm{r}}(t)+\mathrm{i}\psi_{\mathrm{i}}(t) \tag{5.1}$$

we require just

$$1+\kappa_1^2+\kappa_2^2=0. \tag{5.2}$$

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